Note on Linear Differential Equations with Constant Coefficients.

BY F. FRANKLIN.

1. In order to solve the differential equation

$$a_0 \frac{d^m y}{dx^m} + a_1 \frac{d^{m-1} y}{dx^{m-1}} + \ldots . + a_m y \equiv f\left(\frac{d}{dx}\right) y = 0, \tag{1}$$

we may put $y = ve^{\alpha x}$, v being an undetermined function of x, as does M. Picard in his Traité d'Analyse, vol. III, p. 392. The result of this substitution in the first member of the equation is

$$e^{\alpha x} \left[f(\alpha) v + f'(\alpha) \frac{dv}{dx} + \ldots . + f^{(p)}(\alpha) \frac{d^p v}{dx^p} + \ldots . \right], \tag{2}$$

and hence, if α is a p-fold root of the algebraic equation $f(z) = 0$, it is plain that the given differential equation will be satisfied by putting

$$v = C_0 + C_1 x + \ldots . + C_{p-1} x^{p-1}, \tag{3}$$

where the C's are arbitrary constants.

2. The solutions thus obtained, and corresponding to the various roots $\alpha_1, \alpha_2, \ldots . \alpha_n$ of the equation $f(z) = 0$, appear to constitute a complete system of solutions of the differential equation, since they comprise m particular solutions of the type $x^q e^{\alpha x}$; but in order to prove that the system is really complete, it is necessary to show that there can exist no linear relation of the form

$$P_1 e^{\alpha_1 x} + P_2 e^{\alpha_2 x} + \ldots . + P_n e^{\alpha_n x} \equiv 0, \tag{4}$$

where the P's are polynomials in x. M. Picard proves this in a manner sufficiently elementary, but somewhat artificial and involving considerable detail. The impossibility of the relation (4) may, however, be shown instantaneously and without any calculation.

3. We have only to observe that if v be a polynomial of the degree $p-1$, so that $v, \frac{dv}{dx}, \ldots$ are polynomials of descending degrees, the expression (2) does not vanish unless $f(\alpha) = f'(\alpha) = \ldots = f^{(p-1)}(\alpha) = 0$, i. e. unless α is a p-fold root of $f(z) = 0$. Hence, in order that $y = Pe^{\alpha x}$ be a solution of a differential equation (linear and with constant coefficients), P being a polynomial of degree $p-1$, it is *necessary* as well as sufficient that α be a p-fold root of the auxiliary algebraic equation of that differential equation. Now $y = P_2 e^{\alpha_2 x} + \ldots + P_n e^{\alpha_n x}$ is evidently a solution of a certain differential equation whose auxiliary algebraic equation does not possess the root α_1; but, by what has just been said, $y = P_1 e^{\alpha_1 x}$ cannot be a solution of this last-named equation, and therefore cannot be identical with $-(P_2 e^{\alpha_2 x} + \ldots + P_n e^{\alpha_n x})$, so that the relation (4) cannot exist. Q. E. D.